J. Fluid Mech. (1981), *vol.* 112, *pp.* 433–441
Printed in Great Britain

Unsteady flow past a sphere at low Reynolds number

By TAKAO SANO
Department of Mechanical Engineering, University of Osaka Prefecture,
Sakai, Osaka, Japan

(Received 9 September 1980)

This paper complements an earlier paper by Bentwich & Miloh in which the matched asymptotic expansion type of solution is presented for an unsteady low-Reynolds-number flow past a sphere when a constant rectilinear velocity is suddenly imparted to the sphere. It is shown that the matching procedure proposed in the earlier paper is incomplete. The present paper represents a complete procedure for successful matching; the drag of the sphere is calculated up to the term of $O(Re^2 \ln Re)$ using the new procedure.

1. Introduction

Recently, Bentwich & Miloh (1978) considered the problem of unsteady viscous incompressible flow past a solid sphere when a finite rectilinear velocity U is suddenly imparted to the sphere. They obtained the asymptotic solution for small Reynolds number by using the method of matched asymptotic expansions. The matching procedure proposed by them is based on a generalization of the well-known work of Proudman & Pearson (1957) for steady motion. They divided the (r, t) plane (r and t being a non-dimensional radial co-ordinate and the non-dimensional time respectively) into two regions as shown in figure 1; one is the L-shaped region adjacent to r and t axes, where $r = O(1)$ and $t = O(1)$, and the other is the rectangular region far from the axes, where $r = O(Re^{-1})$ and $t = O(Re^{-2})$, where Re denotes the Reynolds number. They proposed to develop separate, locally valid, expansions in terms of Re in these two regions and obtained the first few terms in each of these by adopting the matching procedure demonstrated in figure 1. However, as we shall see below, their method is incomplete for obtaining the higher-order terms.

In the L-shaped region, Bentwich & Miloh introduced the following non-dimensional variables

$$\psi = \psi'/Ua^2, \quad r = r'/a, \quad t = t'\nu/a^2, \tag{1}$$

in terms of which the governing equation can be written as

$$(\partial/\partial t - \Delta^2)\Delta^2\psi = Re\left\{\frac{1}{r^2\sin\theta}\frac{\partial(\psi, \Delta^2\psi)}{\partial(r,\theta)} - \frac{2\Delta^2\psi}{r^3\sin^2\theta}\frac{\partial(\psi, r\sin\theta)}{\partial(r,\theta)}\right\}, \tag{2}$$

where

$$Re = Ua/\nu, \quad \Delta^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \tag{3}$$

In the above equations, (r', θ, ϕ) are spherical co-ordinates with $r' = 0$ at the centre of the sphere and $\theta = 0$ in the direction of the undisturbed stream, a the radius of the

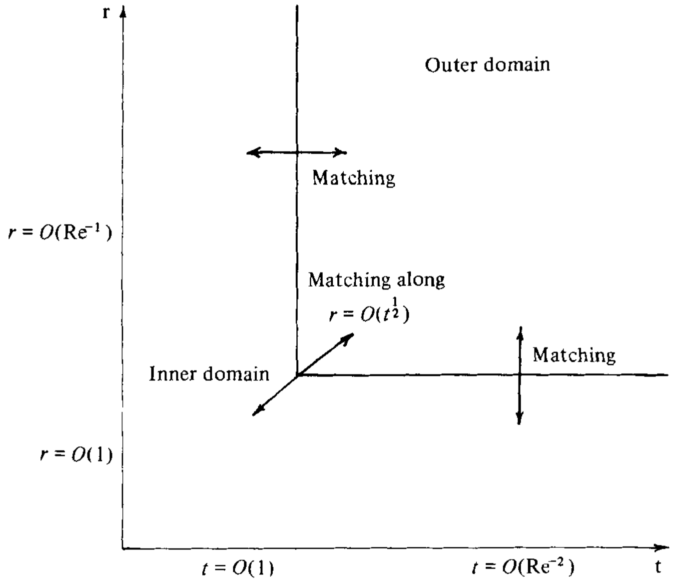

FIGURE 1. Schematic sketch demonstrating the matching procedure proposed by Bentwich & Miloh (1978).

sphere, t' the time, ν the kinematic viscosity and ψ' the stream function related to the velocity components (u'_r, u'_θ) in the (r', θ) directions by

$$u'_r = (1/r'^2 \sin\theta)\, \partial\psi'/\partial\theta, \quad u'_\theta = -(1/r' \sin\theta)\, \partial\psi'/\partial r'. \tag{4}$$

The boundary conditions are

$$\left.\begin{array}{lll} \psi = \partial\psi/\partial r = 0 & \text{for} & r = 1, \quad 0 \leqslant \theta \leqslant \pi, \\ \psi \to \tfrac{1}{2}r^2 \sin^2\theta\, H(t) & \text{as} & r \to \infty, \\ \psi = 0 & \text{for} & t = 0, \quad 1 \leqslant r < \infty, \end{array}\right\} \tag{5}$$

where $H(t)$ is the Heaviside step function. They assumed that the solution of (2) has the form

$$\psi = \psi_0(r, \theta, t) + Re\, \psi_1(r, \theta, t) + \dots. \tag{6}$$

The leading term, ψ_0, satisfies the unsteady Stokes equation. They showed that the equations for ψ_n's have solutions satisfying the boundary condition at infinity as well as the boundary condition on the surface and the initial condition provided that t is finite, and that, as t tends to infinity, the first term ψ_0 approaches the steady Stokes solution. Their work thus seems to imply that the expansion (6) is valid throughout the L-shaped region and that away from this region (6) is singular as in the case of steady motion and there the outer expansion prevails.

We should remember at this point, however, that in the steady motion the solution near the surface is affected by the outer solution which is valid for r of $O(Re^{-1})$ through the matching condition between them. It is therefore probable that the solutions for ψ_n's, which are completely determinable without such a matching process, become invalid as t increases. We can in fact show that the second term in (6), ψ_1, does not approach the corresponding steady solution as $t \to \infty$. The proof is easy. Noting that the solution for ψ_0 has the form $\psi_0 = f(r,t)\sin^2\theta$ (Bentwich & Miloh), we can easily obtain the following equation for ψ_1:

$$(\partial/\partial t - \Delta^2)\,\Delta^2\psi_1 = g(r,t)\sin^2\theta\cos\theta, \tag{7}$$

where f and g are functions of r and t. It is easy to show that the solution of (7) satisfying the initial and boundary conditions (both on the surface and at infinity) has the form

$$\psi_1 = h(r,t)\sin^2\theta\cos\theta, \tag{8}$$

where h is also a function of r and t. On the other hand, the corresponding steady solution given by Proudman & Pearson is

$$\psi_1 = -\tfrac{3}{32}(2r^2 - 3r + 1 - r^{-1} + r^{-2})\sin^2\theta\cos\theta + \tfrac{3}{32}(2r^2 - 3r + r^{-1})\sin^2\theta. \tag{9}$$

Apparently, (8) cannot approach this steady solution as $t \to \infty$, meaning that (6) is invalid for large t even in the vicinity of the sphere, and Bentwich & Miloh's way of dividing the (r,t) plane is correct only if the expansion in the L-shaped region contains one term.

Thus, we have now found that the matching procedure proposed by Bentwich & Miloh is incomplete to obtain the higher-order terms. The purpose of the present paper is to complement their work by representing a complete procedure for successful matching. The drag of the sphere is calculated as far as the term of $O(Re^2 \ln Re)$ using the new matching procedure.

2. Proposed matching procedure

The considerations of the preceding section suggest that (6) is valid only when $t = O(1)$ and therefore that the L-shaped region suggested by Bentwich & Miloh must be subdivided into two domains as shown in figure 2; one is a small-time domain where $t = O(1)$ and the other a large-time inner domain where $t = O(Re^{-2})$ and $r = O(1)$. Thus, including a large-time outer domain where $t = O(Re^{-2})$ and $r = O(Re^{-1})$, the present problem has a three-region structure.† In the small-time domain, the vorticity layer is confined to the inner region near the surface and hence the assumption that the nonlinear inertia terms are negligible is valid throughout the flow field. The solution in this domain is given by (6), which is required to satisfy the boundary conditions both on the surface and at infinity as well as the initial condition. In the large-time inner domain, the $\partial/\partial t$ term is also small, as we shall see later, along with the nonlinear term; namely, the motion in this domain is quasi-static. The requirement of the outer domain in r in the large-time region is due to the fact that, as t increases, the vorticity layer diffuses into the outer region where all the terms in the momentum equation are of the same order of magnitude and the (large-time) inner solution fails.

The construction of the two expansions in the large-time region is made in such a way that: (1) the inner expansion satisfies the boundary condition on the surface; (2) the outer expansion satisfies the boundary condition at infinity; (3) the two expansions match identically in the overlapping domain in space where both expansions are valid and also match the small-time expansion (6) at small values of T, T being a time variable in the large-time region.

† Bentwich & Miloh (1981) have shown independently that a similar structure exists in the circular-cylinder problem.

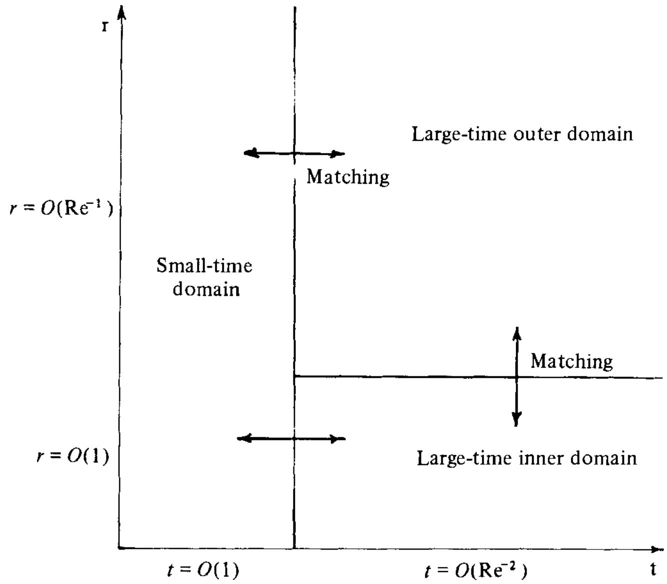

FIGURE 2. Schematic sketch demonstrating the matching procedure proposed in the present paper.

Since the non-dimensional distance over which vorticity diffuses after impulsive start of the sphere is $O(t^{\frac{1}{2}})$, the time required for the vorticity layer to reach to the outer domain, where $r = O(Re^{-1})$, is $O(Re^{-2})$. This suggests that the time variable which is appropriate in the large-time region is

$$T = Re^2 t, \tag{10}$$

which was already introduced by Bentwich & Miloh. In terms of (10), the governing equation can be written as

$$(Re^2\,\partial/\partial T - \Delta^2)\,\Delta^2\psi^{(i)} = Re\left\{\frac{1}{r^2\sin\theta}\frac{\partial(\psi^{(i)}, \Delta^2\psi^{(i)})}{\partial(r,\theta)} - \frac{2\Delta^2\psi^{(i)}}{r^3\sin^2\theta}\frac{\partial(\psi^{(i)}, r\sin\theta)}{\partial(r,\theta)}\right\}, \tag{11}$$

where

$$\psi^{(i)}(r,\theta,T) = \psi(r,\theta,t). \tag{12}$$

The relation (11) is valid in the large-time inner domain. The analysis for the steady flow suggests that the appropriate expansion for $\psi^{(i)}$ is

$$\psi^{(i)} = \psi_0^{(i)}(r,\theta,T) + Re\,\psi_1^{(i)}(r,\theta,T) + Re^2(\ln Re)\psi_2^{(i)}(r,\theta,T) + O(Re^2). \tag{13}$$

In the large-time outer domain, we introduce the following outer variables

$$R = Re\,r, \quad \psi^{(0)}(R,\theta,T) = Re^2\psi(r,\theta,t), \tag{14}$$

in terms of which the governing equation can be written as

$$\left(\frac{\partial}{\partial T} - D^2\right)D^2\psi^{(0)} = \frac{1}{R^2\sin\theta}\frac{\partial(\psi^{(0)}, D^2\psi^{(0)})}{\partial(R,\theta)} - \frac{2D^2\psi^{(0)}}{R^3\sin^2\theta}\frac{\partial(\psi^{(0)}, R\sin\theta)}{\partial(R,\theta)}, \tag{15}$$

where D^2 is the same operator as Δ^2 but with r replaced by R. The appropriate expansion for $\psi^{(0)}$ is

$$\psi^{(0)} = \tfrac{1}{2}R^2\sin^2\theta + Re\,\psi_1^{(0)}(R,\theta,T) + O(Re^2), \tag{16}$$

the first term of which represents a uniform flow.

Note that in the limit $Re \to 0$ the unsteady term in equation (11) can be neglected.

It follows that long after motion has begun the flow field close to the sphere is quasi-steady. In other words, time plays no role in the equation governing the motion in that field and its time dependence is determined by the outer field via matching. The time dependence so obtained is verified later by the fact that the late inner expansion matches that prevailing when motion just begins.

Substituting (13) into (11), we have

$$\Delta^4 \psi_0^{(i)} = 0, \tag{17}$$

$$\Delta^4 \psi_1^{(i)} = \frac{1}{r^2 \sin\theta} \frac{\partial(\psi_0^{(i)}, \Delta^2\psi_0^{(i)})}{\partial(r,\theta)} + \frac{2\Delta^2\psi_0^{(i)}}{r^3 \sin^2\theta} \frac{\partial(\psi_0^{(i)}, r\sin\theta)}{\partial(r,\theta)}, \tag{18}$$

$$\Delta^4 \psi_2^{(i)} = 0. \tag{19}$$

The solution of (17) which matches (16) is clearly the steady Stokes solution

$$\psi_0^{(i)} = \left(\tfrac{1}{2}r^2 - \tfrac{3}{4}r + \frac{1}{4r}\right)\sin^2\theta, \tag{20}$$

and hence (18) becomes

$$\Delta^4 \psi_1^{(i)} = -\frac{9}{4}\left(\frac{2}{r^2} - \frac{3}{r^3} + \frac{1}{r^5}\right)\sin^2\theta\cos\theta. \tag{21}$$

The solution of (21) satisfying the boundary condition on the surface is

$$\psi_1^{(i)} = -\frac{3}{32}\left(2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right)\sin^2\theta\cos\theta + C(T)\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta, \tag{22}$$

where $C(T)$ is an integration constant depending on T and is to be determined through matching with the outer solution.

The equation for the second term $\psi_1^{(0)}$ in the outer expansion (16) is the unsteady Oseen equation and a solution which matches to both the small-time and Stokes solutions has been obtained by Bentwich & Miloh. According to their solution, the asymptotic behaviour of $\psi_1^{(0)}$ for small R is given by

$$\psi_1^{(0)} \sim -\frac{3\sin^2\theta}{4}R + \frac{3\sin^2\theta}{16}\left\{\left(1+\frac{4}{T^2}\right)\operatorname{erf}(\tfrac{1}{2}\sqrt{T}) + \frac{2}{(\pi T)^{\frac{1}{2}}}\left(1-\frac{2}{T}\right)\exp(-\tfrac{1}{4}T) - \cos\theta\right\}R^2 + O(R^3). \tag{23}$$

Therefore, in order to satisfy the matching condition between the inner and outer expansions, the asymptotic behaviour of $\psi_1^{(i)}$ for large r should be of the form

$$\psi_1^{(i)} \sim \frac{3\sin^2\theta}{16}\left\{\left(1+\frac{4}{T^2}\right)\operatorname{erf}(\tfrac{1}{2}\sqrt{T}) + \frac{2}{(\pi T)^{\frac{1}{2}}}\left(1-\frac{2}{T}\right)\exp(-\tfrac{1}{4}T) - \cos\theta\right\}r^2 + \dots. \tag{24}$$

From (22) and (24), $C(T)$ can be determined as

$$C(T) = \frac{3}{32}\left\{\left(1+\frac{4}{T^2}\right)\operatorname{erf}(\tfrac{1}{2}\sqrt{T}) + \frac{2}{(\pi T)^{\frac{1}{2}}}\left(1-\frac{2}{T}\right)\exp(-\tfrac{1}{4}T)\right\}. \tag{25}$$

Thus, $\psi_1^{(i)}$ has been determined as follows:

$$\psi_1^{(i)} = -\frac{3}{32}\left(2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right)\sin^2\theta\cos\theta + \frac{3}{32}\left\{\left(1+\frac{4}{T^2}\right)\operatorname{erf}(\tfrac{1}{2}\sqrt{T}) + \frac{2}{(\pi T)^{\frac{1}{2}}}\left(1-\frac{2}{T}\right)\exp(-\tfrac{1}{4}T)\right\}\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta. \tag{26}$$

For $T \to \infty$, (26) gives its steady counterpart (9) given by Proudman & Pearson. Furthermore, it can easily be shown that the first two terms in the large-time inner expansion (13) match the small-time expansion. If we substitute (20) and (26) into (13) and recast the latter in terms of t, we can obtain the following equation

$$\psi^{(i)} \sim \left\{\left(\tfrac{1}{2}r^2 - \tfrac{3}{4}r + \frac{1}{4r}\right) + \frac{1}{4(\pi t)^{\frac{1}{2}}}\left(2r^2 - 3r + \frac{1}{r}\right) + \ldots\right\}\sin^2\theta$$
$$+ Re\left\{-\frac{3}{32}\left(2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right) + \ldots\right\}\sin^2\theta\cos\theta + O(Re^2 \ln Re). \quad (27)$$

We can easily verify that the first term agrees completely with the asymptotic behaviour of ψ_0 for large t. Moreover, we can see that the θ dependence of the second term and that of the solution for ψ_1, as given by (8), are identical. This suggests that these two also match.

Finally, the solution of (19) can be obtained using the condition that there is no term of $O(Re^2 \ln Re)$ in the large-time outer expansion (Proudman & Pearson) and is found to be given by the following steady solution

$$\psi_2^{(i)} = \frac{9}{160}\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta. \quad (28)$$

The fact that (28) is independent of T suggests that this term matches that of $O(Re^2)$ in the small-time expansion.

3. Discussions

In the large-time region, the inner solution has been obtained up to the term of $O(Re^2 \ln Re)$. Therefore, it is desirable to obtain the second term ψ_1 in the small-time expansion in order to make the present analysis complete. However, since the calculation for obtaining ψ_1 is very tedious and since the term contributes nothing to the drag because of symmetry, it is excluded.

From (20), (26) and (28), the drag of the sphere for the large-time domain can be calculated as

$$D = D_s\left[1 + \tfrac{3}{8}Re\left\{\left(1 + \frac{4}{T^2}\right)\operatorname{erf}\left(\tfrac{1}{2}\sqrt{T}\right) + \frac{2}{(\pi T)^{\frac{1}{2}}}\left(1 - \frac{2}{T}\right)\exp\left(-\tfrac{1}{4}T\right)\right\}\right.$$
$$\left. + \tfrac{9}{40}Re^2 \ln Re + O(Re^2)\right], \quad (29)$$

where D_s denotes the steady Stokes drag, and that for the small-time domain can be written as

$$D = D_s\{H(t) + \tfrac{1}{3}\delta(t) + (\pi t)^{-\frac{1}{2}} + O(Re^2)\}, \quad (30)$$

where the terms of $O(1)$ were obtained by Bentwich & Miloh, $\delta(t)$ denoting the Dirac delta function. From (29) and (30), we can construct a single composite expansion for the drag which is uniformly valid for all values of time by adding them and then subtracting the common part. The result is

$$D = D_s\left[H(t) + \tfrac{1}{3}\delta(t) + (\pi t)^{-\frac{1}{2}} + \tfrac{3}{8}Re\left\{\left(1 + \frac{4}{Re^4 t^2}\right)\operatorname{erf}\left(\tfrac{1}{2}Re\sqrt{t}\right)\right.\right.$$
$$\left.\left. + \frac{2}{(\pi t)^{\frac{1}{2}}Re}\left(1 - \frac{2}{Re^2 t}\right)\exp\left(-\tfrac{1}{4}Re^2 t\right) - \frac{8}{3(\pi t)^{\frac{1}{2}}Re}\right\} + \tfrac{9}{40}Re^2 \ln Re + O(Re^2)\right]. \quad (31)$$

Figure 3 shows the relation between D/D_s and T for several values of Re.

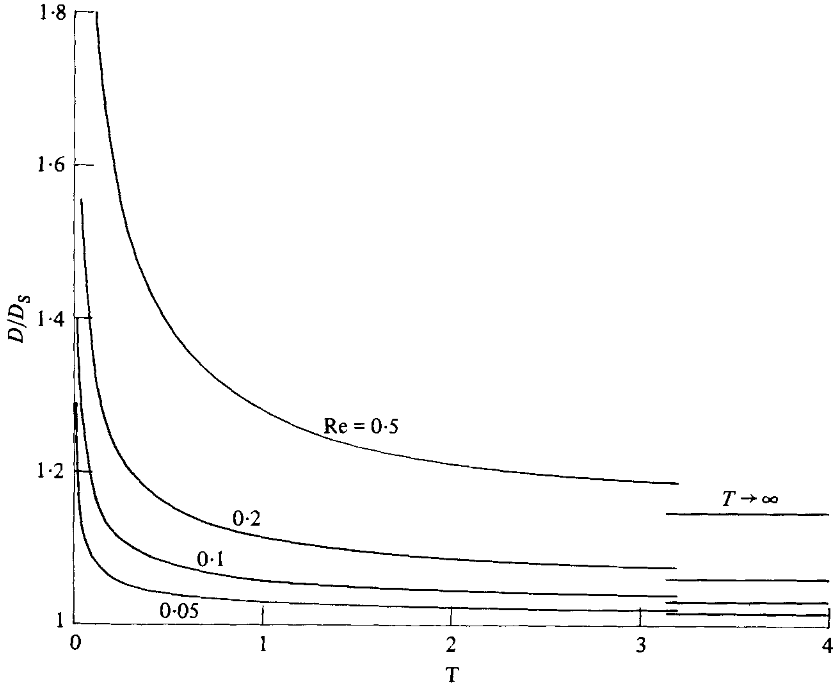

FIGURE 3. Timewise variation in D/D_s.

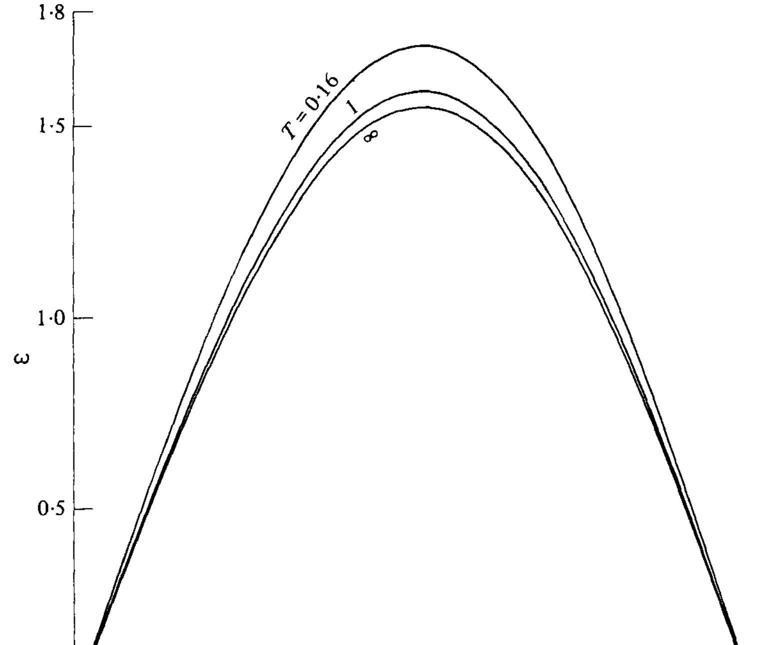

FIGURE 4. Development with time of vorticity on the surface of the sphere for $Re = 0{\cdot}1$.

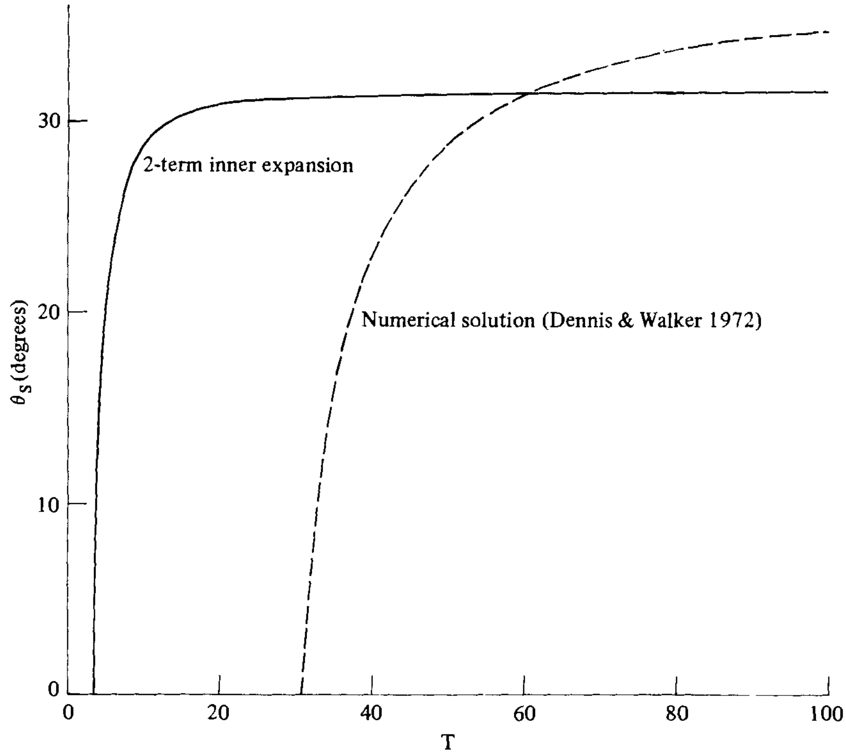

FIGURE 5. Comparison of progression of separation angle θ_s with the numerical solution of Dennis & Walker (1972), at $Re = 20$.

In figure 4 the development with time of the vorticity ω on the surface of the sphere calculated from the large-time inner solution is shown for $Re = 0{\cdot}1$.

Finally, we shall discuss the formation of an eddy behind the sphere, the boundary of which may be calculated from the equation obtained by equating to zero the 2-term large-time inner expansion, $\psi_0^{(i)} + \psi_1^{(i)} Re$. In the similar calculation for steady flow, the eddy first appears at the rear stagnation point when $Re = 8$, which is so large that one would not have expected the low-Reynolds-number expansion to have any validity. Nevertheless, Van Dyke (1975) shows that up to $Re = 60$ the length of the steady eddy calculated from the 2-term inner expansion agrees well with the experimental observations of Taneda (1956). In order to see whether such agreement exists in the unsteady flow also, we compare in figure 5 the progression with time of the separation angle θ_s calculated from (20) and (26) with that obtained numerically by Dennis & Walker (1972), at $Re = 20$. It is seen that agreement is not satisfactory, especially in the value of the separation time T_s, the time at which the eddy first appears. The value of T_s predicted by the present theory is too small compared with that of numerical calculation. As T increases, however, agreement between the two values of θ_s becomes satisfactory. In conclusion, we can say that for the unsteady flow the 2-term (large-time) inner expansion can give information about the eddy only in the final stage near the steady state.

The author wishes to thank the referees for their valuable comments which helped him to improve the presentation.

REFERENCES

BENTWICH, M. & MILOH, T. 1978 *J. Fluid Mech.* **88**, 17.
BENTWICH, M. & MILOH, T. 1981 *J. Engng Math.* (to be published).
DENNIS, S. C. R. & WALKER, J. D. A. 1972 *Phys. Fluids* **15**, 517.
PROUDMAN, I. A. & PEARSON, J. R. A. 1957 *J. Fluid Mech.* **2**, 237.
TANEDA, S. 1956 *Rep. Res. Inst. Appl. Mech. Kyushu Univ.* **4**, 99.
VAN DYKE, M. 1975 *Perturbation Methods in Fluid Mechanics*. Parabolic Press.